

Index of 6-L: One and Two Continuous Samples

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PRACTICAL INFORMATION

Today's lecture:

- follow-up/review from Session 5 based on Exercise 6.95 and 5L–19,
- inference for continuous data, **without assumption of known σ** :
 - * a new distribution: the ***t*-distribution(s)**,¹
- lots of examples (data sets!) and practice with statistical inference,
- **inference for one and two samples** (continuous data):²
 - * a single sample (no assumption of known σ),
 - * two independent samples,
 - * two dependent (paired) samples.

Scheduling notes:

- home assignment is due anytime today (my mailbox/office, or submitted electronically on Moodle); next home assignment will be posted on February 26,
- midterm and final exam dates set (listed at webpage),
- we're heading into the winter semester break (next class on February 23).

¹ PSLS 4e: Chapter 17; S: Section 7.4; IPS 7e: Section 7.1.

² PSLS 4e: Chapters 17-18; S: Chapters 8-9 (parts); IPS 7e: Sections 7.1-2.

OUTLINE OF STATISTICAL ANALYSIS (REVISITED)

- Data description,
- Statistical model,
- Estimation of model's unknown parameter(s),
 - * incl. confidence intervals and/or standard errors,³
- Model check:
 - * comparison of the observed distribution and assumed theoretical distribution (using estimated parameters),
 - * **methods**: graphical (plots) or numerical (tests),
 - * if model is deemed unsatisfactory, **start over with new model**,
- Hypothesis testing:
 - * set up **null hypothesis** H_0 (model simplification) and **alternative hypothesis** H_a ,
 - * **test statistic** and associated **P -value** summarize our confidence **against null hypothesis**, which we may **reject** (low P) or **not reject** (high P),
- Conclusion / Presentation:
 - * summary of test results,
 - * illustrations of the **implications** of the final model, e.g. prediction.

³ Recall, that the standard error (SE) is the standard deviation in the distribution of the estimate, and thus an indication of the estimate's precision.

1-SAMPLE ESTIMATION

Data: sample X_1, \dots, X_n of size n from some distribution with **unknown mean** μ and **unknown standard deviation** σ (and variance σ^2). More specifically, we assume

- the X 's are i.i.d. (independent, identically distributed),
- $EX_i = \mu$ and $\text{sd}X_i = \sigma$ for all X 's.

For **estimation of σ** we use the sample standard deviation:

$$\hat{\sigma} = s \quad (= \sqrt{s^2}) \quad \text{and} \quad \hat{\sigma}^2 = s^2 = \sum_{i=1}^n (X_i - \bar{X})^2 / (n-1),$$

and s^2 is an **unbiased** estimate of σ^2 , explaining why we use $(n-1)$ for s^2 .⁴

Summary of terminology and estimates for a single sample:

Name	Estimate	Parameter	Properties
sample mean	\bar{X}	μ	unbiased
sample variance	s^2	σ^2	unbiased
sample standard deviation	s	σ	biased, natural
(sample variance of mean)	s^2/n	$\sigma_{\bar{X}}^2 = \sigma^2/n$	unbiased
standard error of mean	s/\sqrt{n}	$\sigma_{\bar{X}} = \sigma/\sqrt{n}$	biased, natural

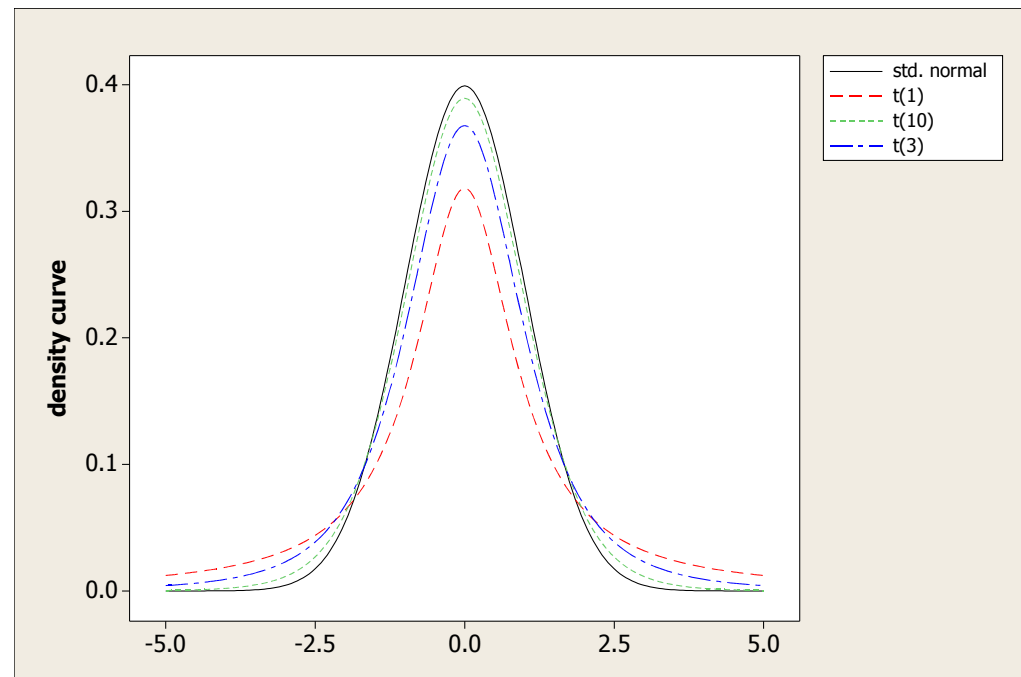
In addition, $\bar{X} \sim N(\mu, \sigma_{\bar{X}})$

- exactly — if the X 's are normally distributed,
- approximately (when n is “large”) — always! (by the CLT)

⁴ We skip over the (small) mathematical calculation showing that s^2 is unbiased.

(“STUDENT’S”) t DISTRIBUTION

- new distribution(s) — to be used **not for modelling** but for inference in a normal distribution model **when σ is estimated from the data**, as the reference distribution for t test statistics (next slide),
- has a **single parameter r** (or “df”):
 - * $r = 1, 2, 3, \dots$
 - * called “**degrees of freedom**” (explanation to follow),
 - * given from the data, and not to be estimated.
- denoted $t(r)$ to indicate degrees of freedom,
- distribution on $(-\infty, \infty) \Rightarrow$ positive and negative values,
- symmetric around zero, almost “bell-shaped” but with heavier tails than $N(0,1)$ (\Rightarrow positive kurtosis),
- when r is large:
 $t(r) \approx N(0, 1)$, see graph.



1-SAMPLE NORMAL DISTRIBUTION INFERENCE

- **Data:** X_1, \dots, X_n (n = number of observations).
- **Model:** observations are a sample (i.i.d.) from $N(\mu, \sigma)$, where μ and σ are unknown parameters.
- **Estimation:** $\hat{\mu} = \bar{X}$ and $\hat{\sigma} = s$.
- **Distribution of estimates:**

$$\hat{\mu} = \bar{X} \sim N(\mu, \sigma/\sqrt{n}), \quad s_{\bar{X}} = s/\sqrt{n},$$

$$(\bar{X} - \mu)/s_{\bar{X}} \sim t(n-1),$$

note that **degrees of freedom (df)** = $n-1$,
- **Confidence interval** with confidence level $1-\alpha$:
$$\mu : \bar{X} \pm t^* s_{\bar{X}} = \bar{X} \pm t^* s/\sqrt{n},$$

where t^* is $(1-\frac{\alpha}{2})$ -percentile of a $t(n-1)$ distribution,⁵
- **Test** of $H_0: \mu = \mu_0$ against alternative H_a :
 - * **test statistic:** $t = (\bar{X} - \mu_0)/s_{\bar{X}} = (\bar{X} - \mu_0)/(s/\sqrt{n})$,
 - * **P-value** from t distribution with $df = n-1$:
 - $H_a: \mu \neq \mu_0$: $P = 2 \times P(t(df) \geq |t_{\text{obs}}|)$,
 - $H_a: \mu > \mu_0$: $P = P(t(df) \geq t_{\text{obs}})$, and $H_a: \mu < \mu_0$: $P = P(t(df) \leq t_{\text{obs}})$,
- note strong **similarities** with z -based procedures.

⁵ In notation, $t^* = t_{1-\alpha/2}(df)$ in the $t(df)$ distribution; see Table C of PSLS, Table 3 of S, Table D of IPS.

EXAMPLE: HUMAN BODY TEMPERATURE

Example 14.9 of PSLS 4e, Example p. 139 of S:

○ **Data:** 130 measurements⁶ of body temperature in °F of healthy adults: X_1, \dots, X_{130} ($n = 130$).

○ **Model:** a sample (i.i.d.) from $N(\mu, \sigma)$.

○ **Estimation:** $\hat{\mu} = \bar{X} = 98.25$ and $\hat{\sigma} = s = 0.733$.

○ **Confidence interval** with confidence level 95% ($\alpha = 0.05$):

$\mu : \bar{X} \pm t^* s_{\bar{X}} = 98.25 \pm 1.98 \times 0.733 / \sqrt{130} = 98.25 \pm 0.13 = (98.12, 98.38)$,
using $t^* = t_{.975}(129) = 1.9785$ from Minitab,

○ **Test** of $H_0: \mu = 98.6$ against alternative $H_a: \mu \neq 98.6$ (“classical” body temp.):

* **test statistic:** $t = \frac{\bar{X} - 98.6}{s_{\bar{X}}} = \frac{98.25 - 98.6}{0.733 / \sqrt{130}} = -5.45$,

* **P-value** from t distribution with $df = n - 1 = 129$:

$$P = 2 \times P(t(129) \geq 5.45) < 0.000001 \text{ (Minitab)}$$

* **Conclusion:** strong evidence to say that average body temperature is different, actually lower, than the “classical” reference value.

⁶ Constructed data for pedagogical purposes (Shoemaker (1996), *Journal of Statistics Education* 4) based on a real study from 1992 whose purpose it was to evaluate the well-established average body temperature of 37.0°C or 98.6°F, Mackowiak et al. (1992), *Journal of the American Medical Association* 268, 1578-1580.

HOW TO FIND PERCENTILES AND *P*-VALUES

Recall that

- the $p\%$ percentile has $p\%$ of the distribution below, and $(100-p)\%$ above, where $(100-p)\%$ is the tail probability,⁷
- *P*-values are typically determined from tail probabilities $P(t \geq t_{\text{obs}})$ in standard distributions, e.g. $N(0, 1)$ or $t(\text{df})$.

Methods to determine percentiles or tail probabilities:

- **Minitab**: Probability Distribution Plot-View Probability menu with Shaded Area defined by Probability or X Value for percentiles and probabilities, respectively,⁸
- **Stata**: functions `normal`, `invnormal`, `ttail`, `invttail`; similar functions in **R**,⁹
- **statistical tables** with values for some confidence/error levels,
 - * what to do, if **df is not in table**? — use largest value below df
⇒ conservative analysis (larger CIs and *P*-values),
 - * what to do, if **t_{obs} -value is not in table**? — find closest (“critical”) values in table, for example $t_1 < t_{\text{obs}} < t_2$, and use the relations

$$P(t \geq t_2) < P(t \geq t_{\text{obs}}) < P(t \geq t_1).$$

⁷ In some statistical tables, the $p\%$ percentile is the **critical value** for a one-tailed test with $\alpha = (100-p)\%$.

⁸ Alternatively, the non-graphical Calc-Probability Distributions menu with Inverse Cumulative Probability or Cumulative Probability, respectively.

⁹ **R** functions: `pnorm`, `qnorm`, `pt`, `qt`.

EXERCISES 7.50 AND 7.52

Exercise 7.50:

Percentiles/critical values for confidence intervals for population mean (with unknown population standard deviation):

- (a) $n = 20$ and $C = 95\%$: $\alpha = 0.05$ and $t^* = t_{1-\alpha/2}(n-1) = t_{.975}(19) = 2.093$.
- (b) $n = 30$ and $C = 90\%$: $\alpha = 0.10$ and $t^* = t_{1-\alpha/2}(n-1) = t_{.95}(29) = 1.699$.
- (c) $n = 50$ and $C = 80\%$: $\alpha = 0.20$ and $t^* = t_{1-\alpha/2}(n-1) = t_{.90}(49) \approx t_{.90}(40) = 1.303$,
— a conservative value (exact value (software): 1.299).

Exercise 7.52:

Testing $H_0: \mu = 0$ against $H_a: \mu > 0$ based on a sample of 15 observations. Observed t -value is $t_{\text{obs}} = 2.15$.

- (a) **degrees of freedom** = $15 - 1 = 14$.
- (b-d) **percentiles** from $t(14)$:

$$t_{0.975}(14) = 2.145 < 2.15 < 2.264 = t_{0.98}(14),$$

with **right-tail probabilities** of 0.025 and 0.02, respectively; therefore,
for $P = P(t(14) \geq t_{\text{obs}})$ we have: **$0.02 < P < 0.025$** ,

- (f) the test is **significant** at the 5% level, but **not significant** at the 1% level.
- (g) **exact P -value** (software) is $P = 0.02476$.

INFERENCE FOR NON-NORMAL DATA

If data show strong / moderate **deviations from normality**:

- **remove outliers** (if any), and see if it helps,
- try to **transform the data**, and see if situation is better for transformed data,
 - * **many transformations exist**: log and square-root common for right-skewed data,
 - * results at transformed scale should always be **backtransformed** to original scale:
 - backtransformed means \sim **medians** in original data,¹⁰
 - for CI's: backtransform both endpoints,
- **nonparametric** statistical methods with no distributional assumptions (next lecture),
- some procedures based on the normal distribution are **robust** (or resistant), that is, work reasonably well even if assumptions are (mildly) violated (saved by the normality of \bar{X} in the CLT!):
 - * difficult to know exactly what is okay and when — some guidelines:¹¹
 - $n < 15$: only if data close to normal (okay!),
 - $15 \leq n < 40$: ok unless strong skewness or outliers,
 - $40 \leq n$: also ok for clearly skewed distributions, but beware of strong outliers.

¹⁰ It is more difficult to get means and SEs on original scale, and potentially less meaningful.

¹¹ Guidelines for t -distribution procedures from PSLS/IPS texts; the discussion in S is less detailed/satisfactory: assume normality or $n > 30$, in my view a very debatable guideline!. If σ is known, the inference is even less affected by non-normality, because it is the procedures involving s^2 that rely most strongly on the normality assumption.

2 PAIRED SAMPLES

Paired (matched, correlated) samples/observations:

- **Data:** $(X_1, Y_1) \dots, (X_n, Y_n)$ independent observation pairs:
 - * typical **examples of pairs:**
 - **same individual:** left–right, before–after,
 - **different individuals:** twins, related or similar individuals,
 - * in **experimental design terminology:**
 - pairs \sim blocks (of size 2),
 - observations within pairs \sim different treatments,
 - * **purpose of pairs:** reduce variability and impact of other (lurking) factors,
- **Model and Analysis:**
 - * usually work with **differences:** $D_i = Y_i - X_i$, (ratios Y_i/X_i or other functions also possible),
 - * assume D_1, \dots, D_n sample from distribution (μ_D, σ_D) , where
$$\mu_D = ED_i = EY_i - EX_i,$$
 - * **hypothesis H_0 :** $\mu_D = 0 \sim$ no difference between (means of) X s and Y s,
 - * **all methods for single sample inference apply!**

2 PAIRED SAMPLES: VISUAL RECEPTIVE FIELD

- **Data:** Neural activity (# spikes/sec) for a monkey's neuron in 9 recordings of both Response (R) and Spontaneous activity (SA), (PSLS Example 27.6)

Recording (i)	SA (X_i)	R (Y_i)	Difference (D_i)
1	2.5	16.7	14.2
2	7.5	20.0	12.5
...
9	17.5	10.0	-7.5

- **Model:** one sample (i.i.d.) of differences D_1, \dots, D_9 assumed to follow $N(\mu_D, \sigma_D)$, where $\mu_D = \mu_Y - \mu_X$ is the parameter of principal interest,
- **Estimation:** $\hat{\mu}_D = \bar{D} = 16.87$, $\hat{\sigma}_D = s_D = 16.40$,
- **95% Confidence interval** for μ_D :

$$\bar{D} \pm t^* s_D / \sqrt{n} = 16.87 \pm 2.306 \cdot 16.40 / \sqrt{9} = 16.87 \pm 12.61,$$

- **Test** of $H_0: \mu_D = 0$ ($\sim \mu_Y = \mu_X$) against alternative $H_a: \mu_D > 0$ ($\sim \mu_Y > \mu_X$):

- * **test statistic:** $t = \frac{\bar{D} - 0}{s_D / \sqrt{n}} = \frac{16.87}{16.40 / \sqrt{9}} = 3.09$,

- * **P-value** from t distribution with $df=8$: $P = P(t(8) > 3.09) = 0.007$,¹²

- * **conclusion:** clearly significant difference between neural activity at SA and R, and higher R activity.

¹² The $t(8)$ -distribution table has $t_{.99}(8) = 2.896$ and $t_{.995}(8) = 3.355$, from which we would get: $0.005 < P < 0.01$.

2 INDEPENDENT SAMPLES – INTRODUCTION

- **Data:**
 $X_1, \dots, X_{n_1} \sim$ first sample, of size n_1 ,
 $Y_1, \dots, Y_{n_2} \sim$ second sample, of size n_2 .
- **Model:** all observations independent, and the X 's and Y 's are samples from separate distributions,
- typical **example:** treatment and control groups, e.g. study on parasite burdens in Lithuanian calves,
- how to **distinguish from paired samples?**
 - * not necessarily the same number of observations (i.e., maybe $n_1 \neq n_2$),
 - * no relation between X_1 and Y_1 , X_2 and Y_2 , etc.
 - * the X 's are interchangeable (“replications”), and the same for the Y 's.

Overview of inference for mean difference $\mu_1 - \mu_2$, based on normal distributions:

- **assumptions:** normal distributions $N(\mu_1, \sigma_1)$ and $N(\mu_2, \sigma_2)$ for the two samples, with all parameters unknown,
- **slightly different procedures** depending on whether
 - (1) $\sigma_1 \neq \sigma_2$ (general situation¹³).
 - (2) $\sigma_1 = \sigma_2$ (simplest; often unrealistic assumption).¹⁴

¹³ Without a specific assumption about the σ 's: we could have $\sigma_1 = \sigma_2$ or $\sigma_1 \neq \sigma_2$.

¹⁴ Not part of VHM 801 syllabus; PLS and S texts avoid this (“pooled variance”) method.

EXERCISE 7.40

Identify **statistical design** as either (1) single sample, (2) matched pairs (paired sample) or (3) two independent samples:

- (a) **two independent samples**, because different groups of children and only one score from each child; the before versus after element is not part of the data collection,
- (b) **two paired samples**, because two scores are collected from each child, in random order; nor is the before versus after element part of the data collection here,
- (c) **one sample**, because only one sample (of 20 measurements) is taken,
- (d) **two independent samples**, because there is no connection between the measurements taken with the new and old method.

What about a slight variation of the design where 10 samples are taken from the specimen and each is analyzed with both the new and old method?

That would be **two paired samples**.

2 INDEPENDENT SAMPLES – EQUAL VARIANCES

- **Models:** 1st sample: $N(\mu_1, \sigma_1)$, 2nd sample: $N(\mu_2, \sigma_2)$,
- **assume** $\sigma_1 = \sigma_2 = \sigma$, based on judgement¹⁵ or test¹⁶,
- **Estimation of means:** $\hat{\mu}_1 = \bar{X} \sim N(\mu_1, \sigma/\sqrt{n_1})$, $\hat{\mu}_2 = \bar{Y} \sim N(\mu_2, \sigma/\sqrt{n_2})$
- **estimation of σ**
 (from s_1 and s_2 in X - and Y -samples):

$$s^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2} \quad \text{and} \quad \hat{\sigma} = s = \sqrt{s^2},$$
 — “pooled” s^2 : a weighted average of s_1^2 and s_2^2 ,
- **standard error** of mean difference: $s_{\bar{X}-\bar{Y}} = s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$,
- **degrees of freedom:** $df = (n_1 - 1) + (n_2 - 1) = n_1 + n_2 - 2$,
- **Confidence interval** of level $(1 - \alpha)$ for $\mu_1 - \mu_2$:

$$\mu_1 - \mu_2 : \bar{X} - \bar{Y} \pm t^* s \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}, \quad t^* = t_{1-\alpha/2}(df),$$
- **Test of $H_0: \mu_1 = \mu_2$** against altern. H_a , using **test statistic:** $t = (\bar{X} - \bar{Y}) / \left(s \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \right)$,
 - * P -value from t distribution, e.g. computed as:
 $H_a: \mu_1 \neq \mu_2: P = 2 \times P(t(df) > |t_{\text{obs}}|)$, or $H_a: \mu_1 > \mu_2: P = P(t(df) > t_{\text{obs}})$,
- note **similarities** with 1-sample procedures.

¹⁵ PSLS/IPS guideline for assuming equal standard deviations: $s_{\max}/s_{\min} \leq 2$.

¹⁶ Variance tests (especially Bartlett’s test) are overly sensitive to non-normality.

2 INDEPENDENT SAMPLES – GENERAL

Similar procedure – changes in $s_{\bar{X}-\bar{Y}}$ and df:

- no assumption of $\sigma_1 = \sigma_2$: \Rightarrow more general procedure (also for when $\sigma_1 \approx \sigma_2$),
- Estimation of means and standard deviations: separately for each sample: $\bar{X}, s_1, \bar{Y}, s_2$,
- standard error of mean difference: $s_{\bar{X}-\bar{Y}} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$,
- degrees of freedom – two approaches:
 - * conservative (too low/“safe”) df: $\min(n_1 - 1, n_2 - 1)$,
 - * approximate with “terrible” formulas¹⁷, but the approximations are generally considered good,
- Confidence interval of level $(1 - \alpha)$ for $\mu_1 - \mu_2$:

$$\mu_1 - \mu_2 : \bar{X} - \bar{Y} \pm t^* \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}, \quad t^* = t_{1-\alpha/2}(\text{df})$$

- Test of $H_0: \mu_1 = \mu_2$ against altern. H_a , using test statistic: $t = (\bar{X} - \bar{Y}) / \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$,
 - * P -values from t distribution in the same way, e.g.
 - $H_a: \mu_1 \neq \mu_2$: $P = 2 \times P(t(\text{df}) > |t_{\text{obs}}|)$,
 - $H_a: \mu_1 > \mu_2$: $P = P(t(\text{df}) > t_{\text{obs}})$.

¹⁷ Minitab uses Satterthwaite method, Stata/R use Welch method; both ok to use.

2 INDEPENDENT SAMPLES: PARASITE DATA

- **Data:** $n_1 = 10$ and $n_2 = 9$ parasite counts of calves on infected (X) and safe (Y) pasture,
- **Model:** 2 independent samples (i.i.d.) from $N(\mu_1, \sigma_1)$ and $N(\mu_2, \sigma_2)$, respectively,
- **Estimation:** $\hat{\mu}_1 = \bar{X} = 51.2$, $\hat{\sigma}_1 = s_1 = 24.0$, and $\hat{\mu}_2 = \bar{Y} = 23.8$, $\hat{\sigma}_2 = s_2 = 17.6$,
- some difference in estimated standard deviations, so we are **not** going to assume that $\sigma_1 = \sigma_2$,

- **Confidence interval** with confidence level 95%:

$$\begin{aligned}\mu_1 - \mu_2 &: \bar{X} - \bar{Y} \pm t^* \sqrt{s_1^2/n_1 + s_2^2/n_2}, \\ &= 27.4 \pm 2.12 \sqrt{24.0^2/10 + 17.6^2/9} = 27.4 \pm 20.3,\end{aligned}$$

where $t^* = t_{.975}(16) = 2.12$ (df computed by software),

- **Test** of $H_0: \mu_1 = \mu_2$ against alternative $H_a: \mu_1 \neq \mu_2$:

$$\text{* test statistic: } t = \frac{\bar{X} - \bar{Y}}{\sqrt{s_1^2/n_1 + s_2^2/n_2}} = 2.86,$$

- * approximate P -value from t distribution with $df=16$:

$$P = 2 \times P(t(df) > 2.86) = 0.011.$$

- * **conclusion:** significant difference between parasite burdens on infected and safe pastures; that is, parasite levels are lower on safe pasture.

SUMMARY NOTES

Key words and concepts:

- statistical inference for **1 sample** (quantitative outcome) on a normal distribution with **unknown parameters** (mean and standard deviation):
 - * sample standard deviation (s) as estimate of population standard deviation (σ), standard error (for sample mean),
 - * t -distribution, degrees of freedom,
 - * formulae for t -based confidence interval and t -test,
- finding/approximating P -values and critical values (t^*),
- **designs** involving 1 and 2 samples (any distribution):
 - * 1-sample, 2 independent samples, 2 paired (dependent, correlated) samples,
 - * 2 paired samples \rightarrow 1-sample for differences,
- choice of method/assumption for **2 independent normal** distribution samples:
 - * **equal variances assumed**: pooled variance estimate, ratio of standard deviations ≤ 2 rule,
 - * **no variance assumption**: df determined by software.